1. Exercises from 4.3/4.4

PROBLEM 1. *(Folland 4.3.13) An example of a non-integrable function of two variables*

- Theorem 4.21 says that if S is measurable, and $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded, and discontinuous on a set with zero content, then f is integrable on S.
- This example is intended to illustrate that if f is not bounded but all the rest of the conditions are satisfied, f can fail to be integrable
- We define $f : \mathbb{R}^2 \to \mathbb{R}$ by the following:

$$f(x,y) = \begin{cases} y^{-2} & 0 < x < y < 1 \\ -x^{-2} & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Claim 1: Pick any $y_0 \in (0,1)$, then $f(x, y_0)$ is integrable as a function of x.

$$f(x,y_0) = \begin{cases} 0 & x = 0 \\ y_0^{-2} & 0 < x < y_0 \\ 0 & x = y_0 \\ -x^{-2} & y_0 < x < 1 \\ 0 & x = 1 \end{cases}$$

- $f(x, y_0)$ is bounded because $y_0 \neq 0$
- $f(x, y_0)$ is discontinuous at 3 points, which is a set of zero content
- $[0, 1]$ is measurable because its boundary has zero content
- These three facts imply (via Thm. 4.21) that $f(x, y_0)$ is integrable as a function of x
- The same argument shows that $f(x_0, y)$ is integrable as a function of y on each x_0-slice.

Claim 2: $f(x, y)$ is not integrable on S

- If f were integrable on S, then by Fubini's theorem, we could evaluate the double integral of f over S by doing the iterated integrals in either order.
- We will show that the answer of the iterated integral depends on whether we chose to integrate x or y first; this would be a contradiction to Fubini's theorem, so we conclude that f is not integrable.
- If we integrate over x first, then

$$\int_0^1 \left(\int_0^y y^{-2}\, dx + \int_y^1 -x^{-2}\, dx \right) dy = \int_0^1 \frac{1}{y} + 1 - \frac{1}{y}\, dy = 1$$

- If we integrate over y first, then

$$\int_0^1 \left(\int_0^x -x^{-2}\, dy + \int_x^1 y^{-2}\, dy \right) dx = \int_0^1 -\frac{1}{x} - 1 + \frac{1}{x}\, dx = -1$$

- Since the value of the integral depends on the order of integration, $f(x, y)$ must be not integrable on S.

PROBLEM 2. *(Folland 4.4.3) Find the volume of the region S, which is bounded by both the sphere $x^2 + y^2 + z^2 = 1$ and the cylinder $x^2 + y^2 = 1$*

- Draw picture
- Decide on a coordinate system: S is symmetric under rotation about the z-axis, suggests to use cylindrical coordinates

$$dV = r\, dr\, d\theta\, dz$$

- For $z > 0$, the sphere is given by $z = \sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}$. For $z < 0$, we have $z = -\sqrt{4 - r^2}$.
- Find the limits of integration: We will integrate z-first, then r, then θ.
- Fix any value of r, θ in S, then the limits of integration in the z-direction are given where the line parallel to the z-axis intersects the sphere.
- r varies between $(0, 1)$
- θ sweeps out the whole circle, $\theta \in (0, 2\pi)$
- The volume is given by:

$$\begin{aligned} V(S) &= \iiint_S dV = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 r \left(\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz \right) dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 2r\sqrt{4 - r^2}\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 \sqrt{4 - u}\, du\, d\theta \\ &= \int_0^{2\pi} \left[-\frac{2}{3}(4 - u)^{3/2} \right] \Big|_0^1 d\theta \\ &= \int_0^{2\pi} \left(\frac{16}{3} - 2\sqrt{3} \right) d\theta \\ &= 4\pi \left(\frac{8}{3} - \sqrt{3} \right) \end{aligned}$$

Problem 3. *Evaluate* $\iint_S (x + y)^4 (x - y)^{-5}\, dA$ *when* S *is the square* $-1 \le x + y \le 1$, $1 \le x - y \le 3$

- Draw a picture
- We will apply theorem 4.37: Let $u = x + y$ and $v = x - y$. Notice that this actually gives us A^{-1} because we have written $(u, v) = A^{-1}(x, y)$ instead of $(x, y) = A(u, v)$.

$$\det A^{-1} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \Longrightarrow |\det A| = \frac{1}{2}$$

- The region of integration in terms of the new coordinates is given by $u = -1 \ldots 1$ and $v = 1 \ldots 3$
- Now applying Thm. 4.37:

$$\begin{aligned} \iint_S (x + y)^4 (x - y)^{-5}\, dx\, dy &= |\det A| \int_{G^{-1}(S)} u^4 v^{-5}\, du\, dv \\ &= \frac{1}{2} \left(\int_1^3 v^{-5}\, dv \right) \left(\int_{-1}^1 u^4\, du \right) \\ &= \frac{1}{2} \left(-\frac{1}{4} \left(3^{-4} - 1^{-4}\right) \right) \left(\frac{1}{5} \left(1^5 - (-1)^5\right) \right) \\ &= \frac{4}{81} \end{aligned}$$

Problem 4. *(Folland 4.4.13) Let* S *be the region in the first quadrant bounded by the curves* $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$, *and* $x^2 - y^2 = 4$. *Compute* $\iint_S x^2 + y^2\, dA$.

- Draw a picture
- We are going to apply Thm. 4.41. We use a change of coordinates $u = xy$ and $v = x^2 - y^2$.

- First compute the Jacobian of the transformation:

$$J^{-1} = |\det DG^{-1}| = \begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix} = 2(x^2 + y^2)$$

- Therefore,

$$2(x^2 + y^2)\, dx\, dy = du\, dv$$

- We can now compute the integral:

$$\iint\limits_S x^2 + y^2\, dx\, dy = \int_1^4 \int_1^3 \frac{1}{2}\, du\, dv = (3)(2)\frac{1}{2} = 3$$